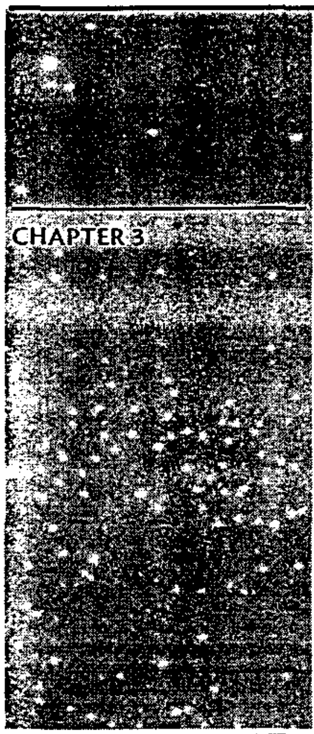

Derivation of the Space-Time Kinetics Equations

AFTER THIS VERY BRIEF REVIEW of the main concepts necessary to the understanding of the physical phenomena we derive here the equations of space-time kinetics. They are made of two partial differential equations, one for the neutron flux, and another for the delayed neutron precursor concentrations. The basic principle that we follow is simply that of the conservation of the number of neutrons and the number of precursor atoms in each hyper volume element of the space and energy domain.

The change in the number of neutrons in a hyper volume element $dVdE$ will be given by the difference between the number of neutrons produced in this hyper volume element, and the number of neutrons disappearing from this same hyper volume element. This is simply expressed as

$$\Delta N = \text{Productions} - \text{Losses}$$

In this same volume element, the atoms of delayed neutron precursor are also produced and destroyed, which gives

$$\Delta C = \text{Productions} - \text{Losses}$$

In the next sections, we will expand all the production and loss terms, which will establish the space-time kinetics equations.

Neutron Conservation

First, let us consider the equations for the neutron flux. We will take into account all interactions that affect the neutron balance in a volume element of the multiplying medium; we look first at neutron production, and then at neutron losses.

Production by Prompt Fission

The total number of neutrons appearing due to fission is given by

$$\int_0^\infty \Sigma_f(\vec{r}, E')\phi(\vec{r}, E', t)dE'$$

Each of these fissions gives rise to ν neutrons on the average, including delayed neutrons which will appear much later. Thus a total of

$$\int_0^\infty \nu\Sigma_f(\vec{r}, E')\phi(\vec{r}, E', t)dE'dVdt$$

neutrons due to fission will appear in the volume element dV, and $(1-\beta)$ multiplied by this number will give the total number of prompt neutrons produced. By multiplying this total number of prompt neutrons by the probability $\chi^p(E)dE$ that a prompt neutron appears in the energy interval from E to $E + dE$, where $\chi^p(E)$ is the prompt neutron spectrum, we find that the average number of prompt neutrons to appear will be

$$\chi^p(E)dE(1-\beta)\int_0^\infty \nu\Sigma_f(\vec{r}, E')\phi(\vec{r}, E', t)dE'dVdt$$

Production by Precursor Disintegration

Each delayed neutron precursor disintegration will give rise to one neutron. But $C_i dV$ is the number of precursor atoms belonging to family i in the volume element dV. By the definition of the disintegration constant, these precursor atoms will give rise to $\lambda_i C_i dVdt$ neutrons in the time interval dt.

By multiplying this number by the delayed neutron spectrum for the family i, $\chi_i^d(E)dE$, and by summing over all precursor families, we obtain the average number of delayed neutrons that appear in the energy interval dE in the volume dV and during the time interval dt,

$$\sum_{i=1}^{D} \chi_i^d(E)dE\lambda_i C_i(\vec{r}, t)dVdt$$

Production by Scattering

Neutrons can also appear in the hyper volume element $dVdE$ by changing energy through collisions with the atoms in the volume element dV.

The number of neutrons in the hyper volume element $dVdE'$ which undergo a scattering reaction in the time interval dt is given by

$$\Sigma_s(\vec{r}, E')\phi(\vec{r}, E', t)dE'dVdt$$

Let $P(E' \rightarrow E)dE$ be the probability that a neutron of energy E' undergoes a collision that brings it in the energy interval between E and $E + dE$. The quantity $P(E' \rightarrow E)$ will be given by the scattering law, and may take more or less complicated expressions according to circumstances. Thus

$$\Sigma_s(\vec{r}, E')\phi(\vec{r}, E', t)dE'dVdtP(E' \rightarrow E)dE$$

represents the number of neutrons in the hyper volume element $dE'dV$ which will appear in the hyper volume element $dEdV$ in the time interval dt. Let us sum over all hyper volume elements $dE'dV$ so as to take into account all neutrons that scatter, which gives

$$\int_0^{\infty} \Sigma_s(\vec{r}, E')P(E' \rightarrow E)\phi(\vec{r}, E', t)dE'dVdtdE$$

The general energy scattering cross-section is denoted by

$$\Sigma_s(\vec{r}, E' \rightarrow E) \equiv \Sigma_s(\vec{r}, E')P(E' \rightarrow E)$$

and we find

$$\int_0^{\infty} \Sigma_s(\vec{r}, E' \rightarrow E)\phi(\vec{r}, E', t)dE'dVdtdE$$

Losses Through Interactions

The total cross-section gives the average number of neutrons that disappear from the hyper volume element $dVdE$ in the time interval dt. This term takes into account losses from absorption and scattering (absorption includes fission). This term is thus

$$\Sigma_t(\vec{r}, E)\phi(\vec{r}, E, t)dVdEdt$$

Losses by Leakage

We consider here the net number of neutrons that leave the volume element dV in the time interval dt. To do this, we use the physical interpretation of the net neutron current. we choose a coordinate system to calculate the leakages through each of the faces of the volume element dV.

In the x direction, we have that the net number of neutrons that leave $dVdE$ through the surface element $dydz$ at position (x, y, z) during the time interval dt is simply

$$-J_x(x, y, z, E, t)dydzdEdt$$

For the neutrons crossing the surface element $dydz$ at position $(x + dx, y, z)$, we will also have

$$J_x(x + dx, y, z, E, t)dydzdEdt$$

The net number of neutrons leaving the volume element by the two surfaces $dydz$ in the time interval dt will be the sum of these two contributions, namely

$$J_x(x + dx, y, z, E, t)dydzdEdt - J_x(x, y, z, E, t)dydzdEdt$$

which becomes

$$\frac{\partial}{\partial x}J_x(x, y, z, E, t)dydzdEdt$$

Identical calculations can be performed in the y and z directions, which gives, after superposing all contributions,

$$\left(\frac{\partial}{\partial x}J_x(\vec{r}, E, t) + \frac{\partial}{\partial y}J_y(\vec{r}, E) + \frac{\partial}{\partial z}J_z(\vec{r}, E, t)\right)dxdydzdEdt$$

We rewrite this in terms of vectors, independent of the coordinate system,

$$\nabla \cdot \vec{J}(\vec{r}, E, t)dVdEdt$$

Neutron Density Change

Finally, the variation of neutron density in the hyper volume element $dVdE$ in the time interval dt will be given by

$$\Delta N(\vec{r}, E, t) = N(\vec{r}, E, t + dt)dVdE - N(\vec{r}, E, t)dVdE$$

which simply becomes

$$\frac{\partial}{\partial t}N(\vec{r}, E, t)dVdt$$

In terms of the neutron flux, this can also be written

$$\frac{1}{v(E)}\frac{\partial}{\partial t}\phi(\vec{r}, E, t)dVdEdt$$

Neutron Flux Equation

By grouping the appropriate terms of the preceding discussion, we obtain the final form of the equation for the neutron flux:

$$\begin{aligned}\frac{1}{v(E)}\frac{\partial}{\partial t}\phi(\vec{r}, E, t) = & -\nabla \cdot \vec{J}(\vec{r}, E, t) - \Sigma_t(\vec{r}, E)\phi(\vec{r}, E, t) \\ & + \int_0^{\infty}\Sigma_s(\vec{r}, E' \rightarrow E)\phi(\vec{r}, E', t)dE' \\ & + \chi^p(E)dE(1 - \beta)\int_0^{\infty}\nu\Sigma_f(\vec{r}, E')\phi(\vec{r}, E', t)dE' \\ & + \sum_{i=1}^{D}\chi_i^d(E)\lambda_i C_i(\vec{r}, t)\end{aligned}$$

Precursor Conservation

We now examine the delayed neutron behavior. The approach will be the same as the one we have taken in the previous section for the flux.

Precursor Production

The number of delayed precursor atoms of family i produced in the volume element dV in time interval dt is the fraction β_i of all neutrons produced by fission in this volume element in this time interval. It will be

$$\beta_i \int_0^\infty \nu\Sigma_f(\vec{r}, E')\phi(\vec{r}, E', t)dE'dVdt$$

Losses of Precursors

The atoms of delayed precursors are lost through beta disintegration. The total number of atoms of delayed precursor i lost in the volume element dV and during the time interval dt is thus

$$\lambda_i C_i(\vec{r}, t)dVdt$$

Precursor Population Change

Finally, the total change in delayed neutron precursor concentration of family i in the volume element is

$$\Delta C_i(\vec{r}, t) = C_i(\vec{r}, t + dt) - C_i(\vec{r}, t)$$

which becomes

$$\Delta C_i(\vec{r}, t) = \frac{\partial}{\partial t}C_i(\vec{r}, t)dVdt$$

Precursor Equation

The final form of the delayed neutron precursor concentration is obtained by grouping together the different terms of the previous section, which gives

$$\frac{\partial}{\partial t}C_i(\vec{r}, t) = \beta_i\int_0^\infty \nu\Sigma_f(\vec{r}, E')\phi(\vec{r}, E', t)dE' - \lambda_i C_i(\vec{r}, t)$$

Final Result

We regroup here for reference the continuous energy form of the space-time kinetics equations

$$\begin{aligned}\frac{1}{v(E)}\frac{\partial}{\partial t}\phi(\vec{r}, E, t) &= -\nabla\cdot\vec{J}(\vec{r}, E, t) - \Sigma_t(\vec{r}, E)\phi(\vec{r}, E, t)\\ &+ \int_0^\infty \Sigma_s(\vec{r}, E' \rightarrow E)\phi(\vec{r}, E', t)dE'\\ &+ \chi^p(E)(1-\beta)\int_0^\infty \nu\Sigma_f(\vec{r}, E')\phi(\vec{r}, E', t)dE'\\ &+ \sum_{i=1}^{D}\chi_i^d(E)\lambda_i C_i(\vec{r}, t)\end{aligned} \qquad \text{(EQ 1)}$$

$$\vec{J}(\vec{r}, E, t) = -D(\vec{r}, E)\vec{\nabla}\phi(\vec{r}, E, t) \qquad \text{(EQ 2)}$$

$$\frac{\partial}{\partial t}C_i(\vec{r}, t) = \beta_i\int_0^\infty \nu\Sigma_f(\vec{r}, E')\phi(\vec{r}, E', t)dE' - \lambda_i C_i(\vec{r}, t) \qquad \text{(EQ 3)}$$